bounded disturbance input.

# Chapter 6 Input-to-State Stability for Stochastic Switched Systems.

## 6.1 Problem Formulation

$$\boxed{\begin{cases} dx(t) = f_{\sigma(t)}(t, x(t), u(t))dt + g_{\sigma(t)}(t, x(t), u(t))\,dW(t). & (6.1a) \\ x(t_0) = x_0. & (6.1b)\end{cases}}$$

$x \in \mathbb{R}^n$ — state vector (right-continuous stochastic process)

$u: [t_0, \infty) \to \mathbb{R}^l$ — input ( $\|u(t)\|_\infty \le 1$ )

$\sigma(t): [t_0, \infty) \to \mathcal{J}$ — switching signal ( $\mathcal{J} = \{1, 2, \cdots, N\}$ )

$\boxed{\text{Markovian switching}}$ : $\Gamma = [\gamma_{ij}]_{N\times N}$ $\quad \gamma_{ij} \ge 0,\ i \ne j.$ $\quad \gamma_{ii} = -\sum_{j=1, j\ne i}^{N} \gamma_{ij}$

$$P\{\sigma(t+h) = j \mid \sigma(t) = i\} = \begin{cases} \gamma_{ij} h + o(h), & i \ne j \\ 1 + \gamma_{ii} h + o(h). & i = j.\end{cases}$$

$\{t_k\}_{k\in\mathbb{N}}$ : switching times. $t_k \nearrow$ and $\lim_{k\to\infty} t_k = \infty$.

$f_i: [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$, $\quad g_i: [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n\times m}$ $\qquad f_i, g_i \in \mathcal{L}_{ad}(\Omega, L^p[t_{k-1}, t_k))$ ($p=1$, $p=2$) $\forall i = 1, \cdots, N.$

to guarantee a unique solution.

$f_i(t, 0, 0) = 0$, $g_i(t, 0, 0) = 0$. $\Rightarrow x \equiv 0$ is a trivial solution.

$x_0$ — initial state ( $\mathcal{F}_0$-measurable. $E[\|x_0\|^p] < \infty$ )

**Lemma 6.1**

Assume $x(t) = x(t; t_0, x_0)$ is the solution to

$$\boxed{dx(t) = f(t, x(t), u(t))dt + g(t, x(t), u(t))dW(t), \quad x(t_0) = x_0}.$$

radially bounded.

$\tau_\infty$ — the explosion time. $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, $\lim_{\|x\|\to\infty} [\inf_{t\ge t_0} V(t, x)] = \infty$.

$\mathcal{L}V(t, x, u) \le \alpha(V(t, x))$. (a.s.) $\qquad \alpha \in \mathcal{K}_c$.

Then $E(V(t, x(t))) \le G^{-1}[G(E(V(t_0, x_0))) + (\ \ - t_0)] < \infty \qquad \forall T \ge t_0$.

where $G(s) = \int_1^s \frac{dt}{\alpha(s)}$. $\quad G^{-1}$ — inverse function of $G$.

Moreover, $x(t)$ is unique and defined for all $t \ge t_0$.

Proof: ① Claim $\tau_\infty = \infty$. (By contradiction)

If not true. $\exists\ \varepsilon > 0$, $T > 0$. s.t. $P\{\tau_\infty \le T\} > \varepsilon$. $\qquad \tau_l \le \tau_\infty$.

Define stopping times $\tau_l\ (l \ge 1)$: $\tau_l = \inf\{t \ge t_0 \mid \|x(t)\| > l\}$ $\Rightarrow \lim_{l\to\infty} \tau_l = \tau_\infty$ (a.s.)

$\Rightarrow P\{\tau_l \le T\} \ge P\{\tau_\infty \le T\} > \varepsilon$.

For all $t \in [t_0, T]$, let $\tau_l(t) = \min\{\tau_l, t\} \Rightarrow \tau_l(t) = \begin{cases} \tau_l, & \tau_l \le t \le T \\ t, & t_0 \le t \le \tau_l\end{cases}$

By Itô formula $\Rightarrow$

(i) For $t \le \tau_l$, $\tau_l(t) = t$, $\forall s \in [t_0, t]$, $\tau_l(s) = s$. $\Rightarrow \int_{t_0}^{\tau_l(t)} \mathcal{L}V(s, x(s), u(s))ds = \int_{t_0}^{t} \mathcal{L}V(\tau_l(s), x(\tau_l(s)), u(\tau_l(s)))ds \le \int_{t_0}^{t} \alpha(V(\tau_l(s), x(\tau_l(s))))ds$

(ii) For $\tau_l \le t \le T$, $\tau_l(t) = \tau_l < t$. $\forall s \in [t_0, \tau_l]$, $\tau_l(s) = s$ $\Rightarrow \int_{t_0}^{\tau_l(t)} \mathcal{L}V(s, x(s), u(s))ds = \int_{t_0}^{\tau_l} \mathcal{L}V(\tau_l(s), x(\tau_l(s)), u(\tau_l(s)))ds$

$$\le \int_{t_0}^{\tau_l} \alpha(V(\tau_l(s), x(\tau_l(s))))ds \le \int_{t_0}^{t} \alpha(V(\tau_l(s), x(\tau_l(s))))ds$$

$$E(V(\tau_l(t), x(\tau_l(t)))) = E(V(t_0, x_0)) + E\int_{t_0}^{\tau_l(t)} \mathcal{L}V(s, x(s), u(s))ds$$

$(*)$
$$\le E(V(t_0, x_0)) + E\int_{t_0}^{t} \alpha(V(\tau_l(s), x(\tau_l(s))))ds$$

$$\le E(V(t_0, x_0)) + \int_{t_0}^{t} \alpha(E(V(\tau_l(s), x(\tau_l(s)))))ds \quad \alpha \text{ is concave}$$

By Bihari's inequality $\Rightarrow E[V(\tau_l(t), x(\tau_l(t)))] \le G^{-1}[G(E[V(t_0, x_0)]) + (t - t_0)]$

$$\le G^{-1}[G(E[V(t_0, x_0)]) + (T - t_0)]$$

On the other hand,

$$E(V(\tau_l(T), x(\tau_l(T)))) = E(V(\tau_l(T), x(\tau_l(T)))\mathbb{1}_{(\tau_l \le T)}) + E[\underbrace{V(\tau_l(T), x(\tau_l(T)))}_{\ge 0}\underbrace{\mathbb{1}_{(\tau_l > T)}}_{\ge 0}]$$

Then $\underset{E(V(\tau_l, x(\tau_l))\mathbb{1}_{(\tau_l \le T)})}{\underset{\parallel}{E(V(\tau_l(T), x(\tau_l(T)))\mathbb{1}_{(\tau_l \le T)})}} \le G^{-1}[G(E[V(t_0, x_0)]) + (T - t_0)]$

Define $\eta_l = \inf\{V(t, x) \mid \|x\| \ge l,\ t \ge t_0\}$. $\|x(\tau_l)\| \ge l$ by the definition of $\tau_l$

$\Rightarrow G^{-1}[G(E[V(t_0, x_0)] + (T - t_0)] \ge E(V(\tau_l, x(\tau_l))\underbrace{\mathbb{1}_{(\tau_l \le T)}}_{\ge 0}) \ge \eta_l \cdot P(\tau_l \le T) > \varepsilon\eta_l$.

Letting $l \to \infty$. since $\lim_{\|x\| \to \infty} \inf_{t \ge t_0} V(t, x) = \infty$ $\Rightarrow \eta_l \to \infty$.

$\Rightarrow$ Contradiction. $\Rightarrow \tau_\infty = \infty$.

② Prove $E[V(t, x(t))] \le G^{-1}[G(E[V(t_0, x_0)] + t - t_0]$

$$E[V(t, x(t))] \le E[V(t_0, x_0)] + E\int_{t_0}^{t} \mathcal{L}V(s, x(s), u(s))ds$$

$$\le E[V(t_0, x_0)] + E\int_{t_0}^{t} \alpha(V(s, x(s)))ds$$

$$\le E[V(t_0, x_0)] + \int_{t_0}^{t} \alpha(E(V(s, x(s))))ds$$

By Bihari's inequality to get the result.

③ local Lipschitz condition $\Rightarrow$ uniqueness.

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Def 6.2.

(i) uniformly asymptotically ISS (aISS) in the pth moment if

$\exists\ \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ s.t. $\forall u$, and $p > 1$.

$$E[\|x(t)\|^p] \le \beta(E[\|x_0\|^p], t - t_0) + \gamma(\|u(t)\|_\infty), \quad \forall t > t_0 \ (t_0 \in \mathbb{R}_+)$$

where $E[\|x(t)\|^p] < \infty$. $x(t) = x(t; t_0, x_0)$ — solution of (6.1).

(ii) exponentially ISS (eISS) in the pth moment if, in addition

$$\beta(E[\|x_0\|^p], t - t_0) \le k E[\|x_0\|^p] e^{-\lambda(t - t_0)} \quad \text{for some } k > 0,\ \lambda > 0.$$

$\mathcal{K} \triangleq \{\alpha \in C([0, \rho); \mathbb{R}_+).\ \nearrow.\ \alpha(0) = 0\}$. $\quad \mathcal{K}_\infty \triangleq \{\alpha \in C([0, \infty); \mathbb{R}_+).\ \nearrow.\ \alpha(0) = 0.\ \alpha(r) \to \infty,\ r \to \infty\}$

$\mathcal{KL} \triangleq \{\beta \in C([0, \rho) \times \mathbb{R}_+; \mathbb{R}_+) : \beta(\cdot, s) \in \mathcal{K},\ \beta(r, \cdot) \searrow,\ \beta(r, s) \to 0\ (s \to \infty)\}$.

## 6.2 Initial-State-Dependent Dwell-time

Switching at fixed times. $\{t_k\}$ all stable modes

Theorem 6.1 (a)ISS under the initial-state dependent on dwell-time condition)

$p\geq 1$. For $i\in \mathcal{S}$, $V_i \in C^{1,2}([t_{k-1}, t_k)\times \mathbb{R}^n; \mathbb{R}_+)$ with $V_i(t,0)=0$. $\forall t\in[t_{k-1},t_k)$

(i) $\exists\ \alpha_{1i}\in \mathcal{K}_\infty$, concave, $\alpha_{2i}\in\mathcal{K}_\infty$. convex. s.t.

$$\alpha_{2i}(\|x\|^p)\leq V_i(t,x)\leq \alpha_{1i}(\|x\|^p) \quad a.s. \quad \Rightarrow \|x\|^p \geq \alpha_{1i}^{-1}(V_i(t,x))$$

(ii) $\exists\ \alpha_{3i}\in\mathcal{K}_v$. $\gamma\in\mathcal{K}$. s.t.

$$\mathcal{L}V_i(t, x(t), u(t)) \leq -\alpha_{3i}(\|x\|^p) \quad a.s.$$

provided that $\|x\|^p > [\alpha_{3i}^{*-1}(\frac{1}{v}\gamma(\|u\|_\infty))] =: \rho_i(\|u\|_\infty)$ (a.s.). where $0<v<1$. $\alpha_{3i}^*(\cdot)=\frac{1}{1-v}\alpha_{3i}(\cdot)$.

(iii) $\forall k\in\mathbb{N}$. $\tau_{isd}$ condition holds: $\tau_{isd}\geq \ln\frac{\theta_{2i}(a_{k-1}E[\|x_0\|^p])}{\theta_{1i}(a_kE[\|x_0\|^p])}$

$a_k \downarrow^+ 0$, $a_0=1$. $\theta_{1i}$, $\theta_{2i}\in\mathcal{K}_\infty$.

$\Rightarrow$ (6.1) is pth moment aISS with ISS-gain

$$\rho_M(\cdot)=\max\{\rho_i(\cdot)=\alpha_{3i}^{*-1}(\gamma^*(\cdot))\mid i\in\mathcal{S}\}. \qquad \gamma^*=\frac{1}{v}\gamma.$$

Proof: ① Define $m_i(t)=E(V_i(t,x(t)))$ $t\in[t_{k-1},t_k)$.

Itô formula + taking expectation + Fubini Theorem + $\begin{cases}\alpha_{1i} & \text{concave}\\ \alpha_{2i} & \text{convex.}\end{cases}$

$$\begin{aligned}\Rightarrow m_i(t) &= m_i(t_{k-1}) + E\int_{t_{k-1}}^t \mathcal{L}V_i(s,x(s),u(s))\,ds\\
&\leq m_i(t_{k-1}) - E\int_{t_{k-1}}^t \alpha_{3i}(\|x(s)\|^p)\,ds\\
&\leq m_i(t_{k-1}) - E\int_{t_{k-1}}^t \alpha_{3i}(\alpha_{1i}^{-1}(V_i(s,x(s))))\,ds\\
&= m_i(t_{k-1}) - \int_{t_{k-1}}^t E[\alpha_i(V_i(s,x(s)))]\,ds\\
&\leq m_i(t_{k-1}) - \int_{t_{k-1}}^t \alpha_i(m_i(s))\,ds\end{aligned}$$

By Bihari's inequality. $G_i(t)=\int_1^t\frac{1}{-\alpha_i(s)}ds$

$$m_i(t)\leq G_i^{-1}[G_i(m_i(t_{k-1}))+(t-t_{k-1})] \triangleq \beta_i^*(m_i(t_{k-1}), t-t_{k-1}). \quad \beta_i^*\in\mathcal{KL}.$$

P.14 By [4.5]. $\exists\ \theta_{1i}^*$, $\theta_{2i}^*\in\mathcal{K}_\infty$ s.t. $\beta_i^*(s,t)\leq\theta_{1i}^*(\theta_{2i}^*(s)\exp(-t))$.

$$\alpha_{2i}(E(\|x(t)\|^p)) \leq m_i(t)\leq \theta_{1i}^*(\theta_{2i}^*(m_i(t_{k-1}))e^{-(t-t_{k-1})}) \leq \theta_{1i}^*(\theta_{2i}^*(\alpha_{1i}(E\|x(t)\|^p))e^{-(t-t_{k-1})})$$

$$\Rightarrow E(\|x(t)\|^p)\leq \alpha_{2i}^{-1}\theta_{1i}^*(\theta_{2i}^*(\alpha_{1i}(E[\|x(t)\|^p]))e^{-(t-t_{k-1})}) \triangleq \theta_{1i}^{-1}(\theta_{2i}(E[\|x(t)\|^p])e^{-(t-t_{k-1})})$$

where $\theta_{1i}(\cdot)=(\theta_{1i}^*)^{-1}(\alpha_{2i}(\cdot))$. $\theta_{2i}(\cdot)=\theta_{2i}^*(\alpha_{1i}(\cdot))$

② For $t\in[t_0,t_1)$. $E[\|x(t)\|^p]\leq\theta_{11}^{-1}[\theta_{21}(E[\|x(t_0)\|^p])e^{-(t-t_0)}]$

first mode By $\tau_{isd}$ condition. at switching time $t=t_1$.

$$E[\|x(t_1)\|^p]\leq\theta_{11}^{-1}[\theta_{21}(E[\|x(t_0)\|^p])e^{-(t_1-t_0)}]\leq a_1E[\|x_0\|^p]$$

$e^{-\tau_{isd}}\theta_{21}(E(\|x_0\|^p))\leq\theta_{11}(a_1E[\|x_0\|^p]) \Rightarrow \leq\theta_{11}^{-1}[\theta_{11}(a_1E[\|x_0\|^p])]$

$$\theta_{22}(a_1 E[\|x_0\|^p]) \le \theta_{12}(a_2 E[\|x_0\|^p])\, e^{\tau_{isd}}$$
$$\Rightarrow \quad \theta_{12}^{-1}\left(\theta_{22}(a_1 E[\|x_0\|^p])\, e^{\tau_{isd}}\right) \le a_2 E[\|x_0\|^p]$$

second mode

For $t \in [t_1, t_2)$. $E[\|x(t_2)\|^p] \le \theta_{12}^{-1}[\theta_{22}(E[\|x(t_1)\|^p])\, e^{-(t_2-t_1)}]$

$\tau_{isd}$ condition $\le \theta_{12}^{-1}[\theta_{22}(a_1 E[\|x_0\|^p])\, e^{-(t_2-t_1)}]$

$\le a_2 E[\|x_0\|^p]$.

For $t \in [t_{k-1}, t_k)$. $E[\|x(t)\|^p] \le \theta_{1i}^{-1}[\theta_{2i}(a_{k-1} E[\|x_0\|^p])\, e^{-(t-t_{k-1})}]$. $\Rightarrow E[\|x(t_k)\|^p] \le a_k E[\|x_0\|^p]$

whenever $\|x(t)\| > [\rho_i(\|u\|_\infty)]^{1/p}$.

Since $\lim_{k\to\infty} a_k = 0$. $\exists\, t$ s.t. $\|x(t)\| = [\rho(\|u\|_\infty)]^{\frac{1}{p}}$.

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**Theorem 6.2.** (stable modes and ustable modes) $J_s$ $J_u$.

$J = J_s \cup J_u$. $V_i \in C^{1,2}([t_{k-1}, t_k) \times \mathbb{R}^n; \mathbb{R}_+)$. $V_i(t, 0) = 0$.

(i) $\forall i \in J$. $\exists\, \alpha_{1i} \in \mathcal{K}_\infty$ concave. $\alpha_{2i} \in \mathcal{K}_\infty$ convex. s.t.

$$\alpha_{2i}(\|x\|^p) \le V_i(t, x) \le \alpha_{1i}(\|x\|^p). \quad (a.s.)$$

(ii) (1) $\forall i \in J_s$. $\exists\, \alpha_{3i} \in \mathcal{K}\nu$. $\rho_i \in \mathcal{K}_\infty$. s.t.

$$\mathcal{L}V_i(t, x, u) \le -\alpha_{3i}(\|x\|^p). \quad (a.s.) \quad \text{whenever} \quad \|x\|^p > \rho_i(\|u\|_\infty).$$

(2) $\forall i \in J_u$. $\exists\, \alpha_{3i} \in \mathcal{K}c$. s.t.

$$\mathcal{L}V_i(t, x, u) \le \alpha_{3i}(\|x\|^p). \quad (a.s.)$$

(iii) $\tau_{isd}$ condition :

(1) $\forall i \in J_s$. $k = 1, 3, 5, \cdots$.

$$t_1 - t_0 \ge \ln \frac{\theta_{21}(E(\|x_0\|^p)}{\theta_{11}(a_1 E[\|x_0\|^p])} > 0$$

$$t_3 - t_2 \ge \ln \frac{\theta_{23}(a_1 A_1 E[\|x_0\|^p])}{\theta_{13}(a_2 E[\|x_0\|^p])} > 0$$

$\cdots$

(2) $\forall i \in J_u$. $k = 2, 4, 6, \cdots$.

$$0 < t_2 - t_1 \le G_2[\alpha_{22}(a_1 A_1 E[\|x_0\|^p])] - G_2[\alpha_{12}(a_1 E[\|x_0\|^p])]$$

$$0 < t_4 - t_3 \le G_4[\alpha_{24}(a_2 A_2 E[\|x_0\|^p])] - G_4[\alpha_{14}(a_2 E[\|x_0\|^p])]$$

$\cdots$

where $0 < a_k < a_k A_k \le a_{k-1}$ with $a_0 = 1$. $\theta_{1i}(\cdot) = \theta_{1i}^{*-1}(\alpha_{1i}(\cdot))$ and $\theta_{2i}(\cdot) = \theta_{2i}^{*}(\cdot) \in \mathcal{K}_\infty$. $G_2, G_4 \cdots$ defined in Lemma 6.1.

$\Rightarrow$ (6.1) is the pth moment a2ISS with the ISS gain $\rho_M(\cdot) \triangleq \max_{i \in J} \rho_i(\cdot)$.

## 6.3. Markovian Switching.

Markovian switching to control the mode switching.

$$(6.7)\begin{cases} dx(t) = f(t, x(t), u(t), \sigma(t))\,dt + g(t, x(t), u(t), \sigma(t))\,dW(t). \\ x(t_0) = x_0, \quad \sigma(t_0) = \sigma_0 \in \mathcal{J}. \end{cases}$$

$V(t, x(t), i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{J}; \mathbb{R}_+)$. $i \in \mathcal{J}$.

$$\mathcal{L}V(t, x(t), u(t), i) = V_t(t, x(t), i) + V_x(t, x(t), i) f(t, x(t), u(t), i) + \tfrac{1}{2}\mathrm{tr}[g^T(t, x(t), u(t), i) V_{xx}(t, x(t), i) g(t, x(t), u(t), i)] + \sum_{j=1}^{N} \gamma_{ij} V(t, x(t), j).$$

### Theorem 6.3 (eISS in pth moment)

For any $i \in \mathcal{J}$, $\forall t \in [t_{k-1}, t_k)$.

(i) $\exists\, k > 0$, $\alpha_i > 0$, $\rho_i \geq 0$, $\sigma_i \geq 0$ s.t.

$\|f(t, x, 0, i)\| \leq k\|x\|$. $\|x^T f(t, x, 0, i)\| \leq \alpha_i \|x\|^2$. (a.s)

$\|g(t, x, 0, i)\| \leq \rho_i \|x\|$. $\|x^T g(t, x, 0, i)\| \leq \sigma_i \|x\|^2$. (a.s)

(ii) $\exists\, \lambda > 0$, $c_1 > 0$, $c_2 > 0$ s.t.

$c_1\|x\|^p \leq V(t, x, i) \leq c_2\|x\|^p$. (a.s.)

$\mathcal{L}V(t, x, u, i) \leq -\lambda\|x\|^p$. (a.s.)

whenever $\|x\| > \rho(\|u\|_\infty)$. $\rho \in \mathcal{K}$. $V \in C^{1,2}([t_0, \infty) \times \mathbb{R}^n \times \mathcal{J}; \mathbb{R}_+)$.

(iii) $f, g$ Locally Lipschitz in $u$, uniformly in $t$, $x$, i.e. $\exists\, c_3, c_4 > 0$ s.t.

$\|f(t, x, u, i) - f(t, x, 0, i)\| \leq c_3\|u\|$. (a.s.)

$\|g(t, x, u, i) - g(t, x, 0, i)\| \leq c_4\|u\|$. (a.s.)

$\Rightarrow$ (6.7) is pth moment eISS, for $0 < p < \min\{2, (3c_4 + 4\sigma_i)/(c_4 + 2\sigma_i)\}$.

Lyapunov exponent $\leq -\lambda/c_2$.

Proof: $i \in \mathcal{J}$, $\beta_i > 0$. define $V(t, x(t), i) = \beta_i \|x(t)\|^p$ — Lyapunov function candidate of ith mode

$$\Rightarrow \mathcal{L}V(t, x, u, i) = p\beta_i \|x\|^{p-2} x^T f(t, x, u, i) + \tfrac{1}{2} p\beta_i \|x\|^{p-2} \|g(t, x, u, i)\|^2 - \tfrac{1}{2}p(2-p)\beta_i \|x\|^{p-4} \|x^T g(t, x, u, i)\|^2 + \sum_{j=1}^{N} \gamma_{ij}\beta_j \|x\|^p.$$

$$\leq -\beta_i^* \|x\|^p + 2M(\|x\|)\|u\|_\infty$$

$$\leq -\lambda^* \|x\|^p + 2M(\|x\|)\|u\|_\infty.$$

where $\beta_i^* = -[\beta_i(p|\alpha_i| + \tfrac{1}{2}p\rho_i^2 + \tfrac{1}{2}p(2-p)\sigma_i^2) + \sum_{j=1}^{N}\gamma_{ij}\beta_j] > 0$

$M(\|x\|) = \max\{\beta_i[c_3 + p\rho_i c_4]\|x\|^{p-1},\ c_4 p\beta_i[-(0.5c_4 + \sigma_i)p + (1.5c_4 + 2\sigma_i)]\|x\|^{p-2}\}$.

To use $\lambda^* \|x\|^p$ to dominate $2M(\|x\|)\|u\|_\infty$.

$$\mathcal{L}V(t,x,u,i) \le -(\lambda^* - \nu)\|x\|^p - \nu\|x\|^p + 2M(\|x\|)\|u\|_\infty . \quad 0<\nu<\lambda^* .$$

$$\le -(\lambda^* - \nu)\|x\|^p = -\lambda\|x\|^p .$$

provided that $\nu\|x\|^p > 2M(\|x\|)\|u\|_\infty$ or.

$\|x\| > 2\beta_i/\nu \cdot [c_3 + p\rho_i c_4]\|u\|_\infty$. if $M(\|x\|) = \beta_i [c_3 + p\rho_i c_4]\|x\|^{p-1}$.

$\|x\| > \{2\beta_i p c_4/\nu \cdot [-(0.5c_4+\sigma_i)p + (1.5c_4+2\sigma_i)]\|u\|_\infty\}^{1/2}$.

if $M(\|x\|) = c_4 p \beta_i [-(0.5c_4+\sigma_i) + (1.5c_4+2\sigma_i)]\|x\|^{p-2}$.

$\overset{(i)}{\Longrightarrow}$ exponential decay